Unified Approach to KdV Modulations

Gennady A. El
Institute of Terrestrial Magnetism, Ionosphere and Radio Wave Propagation, Russian Academy of Sciences, Troitsk, Moscow Region

Alexander L. Krylov
O.Yu. Shmidt Institute of Earth Physics, Russian Academy of Sciences, Moscow

Stephanos Venakides
Duke University, Durham, NC

Abstract We develop a unified approach to integrating the Whitham modulation equations. Our approach is based on the formulation of the initial value problem for the zero dispersion KdV as the steepest descent for the scalar Riemann-Hilbert problem [1] and on the method of generating differentials for the KdV-Whitham hierarchy [2]. By assuming the hyperbolicity of the zero-dispersion limit for the KdV with general initial data, we bypass the inverse scattering transform and produce the symmetric system of algebraic equations describing motion of the modulation parameters plus the system of inequalities determining the number the oscillating phases at any fixed point on the x, t - plane. The resulting system effectively solves the zero dispersion KdV with an arbitrary initial data.

1 Introduction

The initial value problem for the Korteweg – de Vries equation

$$u_t - 6uu_x + \epsilon^2 u_{xxx} = 0, \quad u(x,0) = u_0(x) \tag{1}$$

in the zero dispersion limit $\epsilon \to 0$ has been the subject of intense studies during more than 25 years. The physical interest of this limit is that it allows to model the phenomenon of dispersive shocking in dissipationless dispersive media. In contrast to the usual dissipative hydrodynamics, the regularization of a shock occurs here through the generation of small-scale nonlinear oscillations. Gurevich and Pitaevskii in [3] proposed to describe these oscillations with the aid of the one-phase Whitham modulation equations [4]. Multiphase analog of the Whitham equations was derived by Flaschka, Forest and McLaughlin [5]. Later, Lax and Levermore [6] rigorously showed that the multiphased averaged equations appear in the zero dispersion limit of the initial value problem for the KdV equation with asymptotically reflectionless initial data. For the reflecting potentials the zero dispersion theory was constructed by Venakides [7] who also identified the parameters describing the weak limit of the solution in the Lax-Levermore approach with the Riemann invariants of the modulation equations in [5] (see [8], [9]).

The modulation system itself, however, has to be integrated to reveal dependence of the Riemann invariants (modulation parameters) on x and t. For this, the observation of Tsarev [10] (see also [11]) who generalized the classical hodograph method to a multidimensional (in the space of dependent variables) case was crucial. The result of applying this generalized hodograph transform to the averaged KdV is the overdetermined consistent system of linear PDE's. Tsarev's results were put into the algebro-geometrical setting by Krichever [12], [13].

Each solution to the Tsarev system gives rise to some local solution of the nonlinear modulation system. These solutions generically exist only within definite regions of the x, t - plane. To obtain the global solution one should supplement the constructed local solutions with the information about the number N of nonlinear phases in each region and provide the smooth matching of solutions with different N on the phase transition boundaries. The simplest, yet very important, class of problems with $N \leq 1$ has been investigated in works of Tian [14], [15], Gurevich, Krylov, El and their collaborators [16], [17] who obtained a number of exact solutions for the initial value problems with monotone and hump-like data having the only break point. Tian [15] was also able to prove that the solutions to the Whitham system in this case do globally belong to $N = 1$.

More general local solutions to the modulation system in the case of arbitrary N were constructed in a symmetric form by El [2] with the aid of fundamental solution (an analog of the Green function) to the Tsarev equations.

Some global solutions involving the case $N = 2$ were constructed recently by Grava [18] on the basis of Dubrovin's variational approach to the Whitham equations [19].

The common feature of all pointed methods for constructing the global solutions to the Whitham equations is the necessity to follow all earlier times $t < t_0$ to obtain the solution at t_0. This difficulty has been bypassed resently by Deift, Venakides and Zhou [1] who obtained the algebraic equations for the KdV modulations avoiding the integration of the modulation equations themselves. The method of DVZ is based on reformulation of the initial value problem for the zero dispersion KdV with decaying solitonless analytic initial data as the steepest descent for the scalar Riemann-Hilbert problem for the complex phase function. As a result, they obtain not only the local solutions to the modulation equations, but also the system of inequalities enabling to determine the number of phases locally at each point of the x, t-plane.

It is clear, however, that the requirements of decaying and pure reflection for the initial potential are not essential in the zero dispersion limit. Really, assuming the finite speed of propagation (i.e. validity of the Whitham equations) for the zero-dispersion KdV with an arbitrary initial data one can see that only the finite part of the initial data contributes to the solution at any finite t. In this case, considering the evolution of 'multihump' potential and and then tending number of 'humps' to infinity one arrives at the solution for nondecaying initial data. Certainly, for the finite ϵ the results obtained in this way are valid within finite time interval until the semiclassical (Whitham) spectrum begins to compete with the fine spectrum of the potential. The assumption of hyperbolicity in the limit studied allows also to omit the requirement of analyticity for the initial data.

In this work, we combine methods of Deift, Venakides, and Zhou [1] and El [2] to produce the global solution to the KdV modulation equations with an arbitrary initial data. Namely,

by *assuming* validity of the modulation equations in the case of the zero dispersion KdV with an arbitrary initial data we reformulate the initial value problem for the Whitham system as the Riemann-Hilbert problem and produce the symmetric system of algebraic equations supplemented by the system of inequalities determining both the motion of the Riemann invariants and the change of the genus of the Riemann surface (number of oscillating phases). We represent the local part of the obtained general solution in a potential form with the generalized functional of the Peierls-Fröhlich type [20], [21], [22] as a potential. Recently the functionals of this type in more particular form were shown to give rise by minimization to the global solutions for the Whitham equations with monotone initial data [19].

We also make identification of the part of the results obtained in [1] with the results obtained earlier in [12] , [2] and [19] which unifies different approaches developed independently in this area.

2 Summary of the Riemann-Hilbert Steepest Descent Method for the Zero Dispersion KdV with Decaying Solitonless Initial Data

It is well known that the inverse scattering transform can be reformulated as a matrix Riemann-Hilbert (RH) problem (see [23], [24]). Deift, Venakides, and Zhou [1] showed by considering the quasiclassical asymptotics for the initial value problem (1) in the case of one-hump solitonless initial data $0 \leq u_0(x) \leq 1$ that this problem can be asymptotically (as $\epsilon \to 0$) reduced to the scalar RH problem for the complex phase function $g(\lambda)$. We present here the resulting formulas (for details see [1], [25]).

Let the interval $(0, 1)$ of the real axis on the λ - plane is partitioned into a finite set of intervals $\{G_k(k = 0, 1, \ldots, N) :\ (0, r_1), (r_2, r_3), \ldots, (r_{2N}, r_{2N+1})\}$, $\{B_n(n = 1, \ldots, N + 1) : (r_1, r_2), (r_3, r_4), \ldots, (r_{2N+1}, 1)\}$.

We introduce the combination

$$\alpha(\lambda) = 4t\lambda^{3/2} + x\lambda^{1/2}\,, \tag{2}$$

which will be important hereafter, and the phases of the reflection (from the right) and the transmission coefficients for the scattering on the potential $u_0(x)$.

$$\rho_+(\lambda) = x^+\lambda^{1/2} + \int_{x^+}^{\infty} [\lambda^{1/2} - (\lambda - u_0(x'))^{1/2}]dx'\,, \tag{3}$$

$$\tau(\lambda) = \int_{x^-}^{x^+} (u_0(x') - \lambda)^{1/2} dx'\,. \tag{4}$$

where $x^{\pm} = x^{\pm}(\lambda)$ are the roots of the equation $u_0(x) = \lambda$.
Then the RH problem defining the complex phase function $g(\lambda)$, which plays the central role in the future, takes the form

On the intervals $\{G_k, (k = 0, \ldots, N) : \ (0, r_1), (r_2, r_3), \ldots, (r_{2N}, r_{2N+1})\}$ the following relations hold:

$$\frac{g'_+ + g'_-}{2} = \rho'_+ - \alpha' \tag{5}$$

$$-\tau < \frac{g_+ - g_-}{2i} < 0\,, \tag{6}$$

It follows from the equality (5) that on each interval G_k: $g_+ + g_- - 2\rho_+ + 2\alpha = -\Omega_k$, where Ω_k is some constant of integration; without loss of generality one can put $\Omega_0 = 0$.

On the intervals $\{B_k, (k = 1, \ldots, N) : \ (r_1, r_2), (r_3, r_4), \ldots, (r_{2N-1}, r_{2N})\}$ we have

$$\frac{g_+ - g_-}{2i} = -\tau\,, \tag{7}$$

$$\frac{g'_+ + g'_-}{2} < \rho'_+ - \alpha'\,. \tag{8}$$

The latter ineguality is due to the further reduction of the RH problem with the aid of the steepest descent method [1].

On the remaining interval $\{B_{N+1} : (r_{2N+1}, 1)\}$ there exist two possibilities:

$$\text{case A:} \qquad \frac{g_+ - g_-}{2i} = -\tau\,, \qquad \frac{g'_+ + g'_-}{2} < \rho'_+(\lambda) - \alpha'\,. \tag{9}$$

which coincides with the conditions (7), (8)
or

$$\text{case B:} \qquad g_+ - g_- = 0\,, \qquad \frac{g'_+ + g'_-}{2} > \rho'_+(\lambda) - \alpha'\,. \tag{10}$$

Also we have outside the interval $(0, 1)$

$$g_+ + g_- = 0 \qquad \text{if} \qquad \lambda < 0, \tag{11}$$

$$g_+ - g_- = 0 \qquad \text{if} \qquad \lambda > 1\,. \tag{12}$$

In all above formulas we denoted

$$g_\pm(\lambda) \equiv \lim_{\delta \to 0} g(\lambda \pm i\delta)$$

The additional requirement imposed on the function $g(\lambda)$ is

$$\text{The functions } (\sqrt{\lambda} g'(\lambda))_\pm \ \text{ are continuous for real } \lambda\,, \tag{13}$$

Also it follows from [1] that the following asymptotic holds

$$g(\lambda) = g_1/\lambda^{1/2} + O(1/\lambda) \qquad \text{as } \lambda \to \infty\,. \tag{14}$$

One can see that just formulated RH problem contains not only equations defining the function $g(\lambda)$ on the intervals B_k and G_k but also the inequalities which define the overall number $2N$ of the intervals (if N has been chosen wrongly then at least one of the inequalities will be violated).

We observe that for each fixed x and t, g' also satisfies a scalar RH problem on the real axis. Indeed, $g'_+ + g'_- = 0$ when $\lambda < 0$, and $g'_+ - g'_- = 0$ when $\lambda > 1$. When $0 < \lambda < 1$, the equalities (5), (7), (9) and (10) specify $g'_+ + g'_- - 2\rho' + 2\alpha' = 0$ on each interval G_k, while $g'_+ - g'_-$ either equals 0 or equals $-2i\tau'$ outside these intervals. We only consider $g'_+ - g'_- = -2i\tau'$ when λ lies between any two of the G_k's while on the remaining interval $(r_{2N+1}, 1)$, we examine both possibilities i.e. $g'_+ - g'_- = -2i\tau'$ (case A) or $g'_+ - g'_- = 0$ (case B).

Necessarily $g'(\lambda)$ has the following form,

$$g'(\lambda) = \sqrt{R_{2N+1}(\lambda)} \left(\int_{\cup G_k} \frac{2\rho'(\mu) - 2\alpha'(\mu)}{\sqrt{R^+_{2N+1}(\mu)}(\mu - \lambda)} \frac{d\mu}{2\pi i} + \int_{(0,E)\setminus \cup G_k} \frac{-2i\tau'(\mu)}{\sqrt{R^+_{2N+1}(\mu)}(\mu - \lambda)} \frac{d\mu}{2\pi i}, \right), \tag{15}$$

where

$$R_{2N+1}(\lambda) = \prod_{j=1}^{2N+1} (\lambda - r_j)\,, \tag{16}$$

and $E = 1$ in the case A and $E = r_{2N+1}$ in the case B. Here, $\sqrt{R_{2N+1}(\lambda)}$ is positive for $\lambda > r_{2N+1}$. Also, $\sqrt{R^+_{2N+1}(\lambda)}$ denotes the boundary value from above.

A sufficient number of conditions to determine the endpoints of the intervals G_j can now be written down. Indeed, the condition $g(\lambda) = O(\lambda^{-1/2})$ for large λ implies $g'(\lambda) = O(\lambda^{-3/2})$, which leads to the following moment conditions,

$$\int_{\cup G_j} \frac{\rho'(\lambda) - \alpha'(\lambda)}{\sqrt{R^+_{2N+1}(\lambda)}} \lambda^k d\lambda + \int_{(0,E)\setminus \cup G_j} \frac{-i\tau'(\lambda)}{\sqrt{R^+_{2N+1}(\lambda)}} \lambda^k d\lambda = 0\,, \tag{17}$$
$$k = 0, \dots, N\,.$$

A second set of conditions is obtained by integrating g' around each G_j and using (7): In case A we obtain

$$\int_{G_j} (g'_+(\lambda) - g'_-(\lambda)) d\lambda = -2i(\tau(r_{2j+1}) - \tau(r_{2j})), \quad j = 1, \dots N\,. \tag{18}$$

In case B, $\tau(r_{2N})$ and $\tau(r_{2N+1})$ must be replaced by zero.

Conditions (17) and (18) represent a system of $(N+1) + N = 2N+1$ independent equations for the $2N+1$ unknowns (the branch points $r_1, \dots, r_{2N+1}$ of the Riemann surface (16)). Conversely suppose that for given x, t and some N, the quantities $r_1, \dots, r_{2N+1}$ satisfy conditions (17) and (18), giving rise to an explicit expression (15) for g', and hence for g by integration. Suppose further that the function g so constructed, also satisfies the inequalities (6) – (10). Then g is the desired solution of the scalar RH problem.

Deift, Venakides and Zhou show that the parameters r_j represent the "semiclassical spectrum" of the problem, i.e. they are the branch points of the spectral Riemann surface defining the local finite-gap solution of the KdV. In Sec.4 we shall directly identify the moment conditions (17), (18) with the local solution of the KdV-Whitham system.

Now we present a number of important relations which will be very useful for the identification.

We observe from the relations (5) – (12), that $g(\lambda)$ also satisfies a RH problem. Solving this RH problem in exactly the same way as the problem for g' we obtain similarly to (15)

$$g(\lambda) = \sqrt{R_{2N+1}(\lambda)} \sum_{j=0,\dots,N} \left(\int_{G_j} \frac{2\rho_+(\mu) - 2\alpha(\mu) - \Omega_j}{\sqrt{R^+_{2N+1}(\mu)}(\mu-\lambda)} \frac{d\mu}{2\pi i} + \int_{(0,E)\setminus \cup G_j} \frac{-2i\tau(\mu)}{\sqrt{R^+_{2N+1}(\mu)}(\mu-\lambda)} \frac{d\mu}{2\pi i} \right) . \tag{19}$$

The expression for the constant of integration Ω_j is

$$\Omega_j = -2x \oint_{a_\infty} \lambda^{1/2}\psi_j - 8t \oint_{a_\infty} \lambda^{3/2}\psi_j + 4\int_{\cup G_k} \rho_+\psi_j - 4\int_{(0,E)\setminus\cup G_k} i\tau\psi_j \, . \tag{20}$$

$$\equiv x\Omega_{k1} + t\Omega_{k2} + \Omega_{k3} \quad j=1,\dots,N\,,\ \Omega_0 = 0\,.$$

We recall that

$$E = 1 \quad \text{(case A)} \quad or \quad E = r_{2N+1} \quad \text{(case B)} \tag{21}$$

The basis of holomorphic differentials ψ_j is given by

$$\psi_j = \sum_{k=0}^{N-1} c_{k,j} \frac{\lambda^k}{\sqrt{R_{2N+1}(\mathbf{r},\lambda)}} d\lambda\,, \tag{22}$$

$$\oint_{\alpha_k} \psi_j = \delta_{jk}\,, \quad k,j = 1,\dots,N. \tag{23}$$

The contours α_k $(k = 0,1,\dots,N)$ surround the intervals $(r_2,r_3),\dots,(r_{2j},r_{2j+1}),\dots,(r_{2N},r_{2N+1})$ clockwise;

The identities hold for Ω_{j1} and Ω_{j2}:

$$\Omega_{j1} = -Res_\infty \lambda^{1/2}\psi_j\,, \quad \Omega_{j2} = -4Res_\infty\lambda^{3/2}\psi_j\,, \tag{24}$$

$$\partial_x\Omega_j = \Omega_{j1}\,, \quad \partial_t\Omega_j = \Omega_{j2}\,. \tag{25}$$

One can observe from (24) that in normalization (23) accepted, Ω_{j1} and Ω_{j2} can be identified with the wave number k_j and the frequency ω_j correspondingly [5], where j is the number of the phase , $j = 1,\dots,N$ Really (taking into account the change of the normalization in comparison with [5]), we have the expansion as $\lambda\to\infty$

$$\psi_j = \frac{1}{\lambda^{3/2}}(k_j + \frac{1}{4\lambda}\omega_j + \dots); \tag{26}$$

where

$$k_j = c_{N-1,j}, \quad \omega_j = 2c_{N-1,j}\sum_{m=1}^{2N+1} r_m + 4c_{N-2,j}\,. \tag{27}$$

Taking the mixed derivatives of (25) we arrive at N equations expressing the wave number conservation laws, for each of N phases.

$$\partial_t k_j = \partial_x \omega_j \quad , j = 1, \dots, N. \tag{28}$$

3 Equations for the Branch Points in a Symmetric Form. Identification with Local Solutions to the Whitham Equations

Now we obtain equations for the branch points r_j of the Riemann surface, equivalent to the moment conditions (17),(18), in a symmetric form.

Calculating $\partial_x \Omega_j$ directly from (20) we obtain

$$\partial_x \Omega_j = \Omega_{j1} + \sum_{k=1}^{2N+1} \frac{\partial \Omega_j}{\partial r_k} \partial_x r_k\,, \quad j = 1, \dots, N\,, \tag{29}$$

which, together with (25), implies

$$\sum_{k=1}^{2N+1} \frac{\partial \Omega_i}{\partial r_k} \partial_x r_k = 0$$

for any solution $r_k(x,t)$.

Thus,

$$\partial_j \Omega_i = 0\,, \tag{30}$$

$$\partial_j \equiv \frac{\partial}{\partial r_j}, \quad j = 1, \dots, 2N+1, \quad i = 1, \dots, N\,,$$

provided $\partial_x r_j \neq 0\,, \qquad j = 1, \dots, 2N+1$.

The system (30) is the system of $N(N+1)$ algebraic equations for $2N+1$ variables $r_j(x,t)$. Due to the uniqueness of the solution for the Riemann-Hilbert problem (5) – (12) it has to be equivalent to the moment conditions (17),(18). To show that it is enough to prove the consistency of the system (30). In other words one has to show that all N closed systems for $r_j (j = 1, ..., 2N+1)$ forming the overdetermined system (30) are equivalent which is to say that (30), in fact, does not depend on the index i.

With this aim in view we present the following lemma.

<u>Lemma</u> (Consistency)

The overdetermined system $\partial_j \Omega_i = 0\,,\ j = 1, \dots, 2N+1\,,\ i = 1, \dots, N$, where Ω_i is defined by (20) is consistent and equivalent to the symmetric system of $2N+1$ algebraic equations with respect to $2N+1$ variables r_j:

$$\oint_{a_\infty} \{x\lambda^{1/2} + 4t\lambda^{3/2}\}\Lambda_j = \oint_{\cup\alpha_k} \rho_+(\lambda)\Lambda_j - i \oint_{\cup\beta_n\setminus\beta_E} \tau(\lambda)\Lambda_j \ . \tag{31}$$

The contours $\alpha_k,\ (k = 0, \ldots, N)$ as it was mentioned above surround the intervals G_k clockwise. The contours $\beta_n,\ (n = 1, \ldots, N+1)$ surround the intervals B_n clockwise, and the contour β_E surrounds the interval $(E, 1)$, where $E = 1$ (case A) or $E = r_{2N+1}$ (case B).

The differential $\Lambda_j, \quad j = 1, \ldots, 2N+1$ is defined by

$$\Lambda_j = \frac{\partial_j \psi_i}{\partial_j k_i} = \frac{\lambda^N + \sum\limits_{k=1}^{N} \lambda^{N-k} p_{k,j}}{(\lambda - r_j)\sqrt{R_{2N+1}}} d\lambda \,, \tag{32}$$

and the coefficients $p_{k,j}$ can be found unambigously from the normalization conditions

$$\oint_{\alpha_m} \Lambda_j = 0 \,, \quad m = 1, \ldots, N \,, \tag{33}$$

The proof of the lemma can be found in Appendix 1.

Theorem (Identification) The system of algebraic equations (31) implicitely defining positions of the branch points $r_j(x,t)$ of the hyperelliptic Riemann surface in the zero dispersion limit of the RH problem for the KdV equation satisfies the N-phase averaged Whitham-KdV system [5], [6].

$$\partial_t r_j = V_j(\mathbf{r})\partial_x r_j \,, \quad j = 1, \ldots, 2N+1 \,, \tag{34}$$

where $V_j(\mathbf{r})$ are computed as certain combinations of complete hyperelliptic integrals [5].

Proof

To identify algebraic system (31) with the solution of the Whitham equations (34) we make use of the Tsarev result [10], [11]:

If $W_j(r_1, \ldots, r_{2N+1})$ is a solution of the linear overdetermined consistent system

$$\frac{\partial_i W_j}{W_i - W_j} = \frac{\partial_i V_j}{V_i - V_j} \ ; \tag{35}$$

$$i \neq j; \ i, j = 1, \ldots, 2N+1 \,,$$

then the system of algebraic equations (generalized hodograph transform)

$$x + V_j t = W_j \tag{36}$$

gives implicitely the smooth solution $r_j(x,t)$ to the Whitham system (34) provided $\partial_x r_j \neq 0$.

First, we observe that the following identities hold

$$-\frac{1}{2\pi i}\oint_{a_\infty}\lambda^{1/2}\Lambda_j = 1 \qquad \text{for all } j\,, \tag{37}$$

$$\frac{2}{\pi i}\oint_{a_\infty}\lambda^{3/2}\Lambda_j = V_j\,. \tag{38}$$

The first equality immediately follows from the definition (32). The second one requires a bit more detailed consideration. We make use of the fact that the system (34) implies existence of the wave number conservation law (28). Then, introducing the Riemann invariants r_j into the equation (28) explicitely one easily gets the representation for the characteristic speeds of the Whitham system (34) (see [26], [16] for the case $N = 1$ and [2] for arbitrary N).

$$V_i = \frac{\partial_i \omega_j}{\partial_i k_j} \quad \text{for any } j\,. \tag{39}$$

The expression (39) can be interpreted as a generalization of the group velocity notion to the case of nonlinear waves [2]. It should be noted that the relationships analogous to (39) arise in the classical theory of hyperbolic systems as the compatibility condition providing the existence of an additional conservation law [27]. Using (24), (26), (32) we arrive at

$$V_i = \frac{2}{\pi i}\frac{\partial_i}{\partial_i k_j}\oint_{a_\infty}\psi_j\lambda^{3/2}d\lambda = \frac{2}{\pi i}\oint_{a_\infty}\lambda^{3/2}\Lambda_j = -4p_{1,i}\,. \tag{40}$$

The solution (31) then takes the form (36) provided

$$W_i = -\frac{1}{2\pi i}\{\oint_{\cup\alpha_k}\rho_+(\lambda)\Lambda_i - i\oint_{\cup\beta_n\backslash\beta_E}\tau(\lambda)\Lambda_i\}\,, \tag{41}$$

where the functions $\rho_+(\lambda)\,, \tau(\lambda)$ are supposed to be analytic.

We have to prove, therefore, that (41) does solve the linear system (35) which implies proving that Λ_j does solve this system at any λ. With this aim in view we consider the following combinations occuring in the left-hand part of (35)

$$\partial_i\Lambda_j = \partial_i p_{1,j}\frac{\lambda^N + \sum\limits_{k=1}^{N}\lambda^{N-k}a_{k,i}}{(\lambda - r_i)(\lambda - r_j)\sqrt{R_{2N+1}}}d\lambda\,, \tag{42}$$

where all $a_{k,i}$ are uniquiely defined by the conditions following from the normalization (33)

$$\oint_{\alpha_m}\partial_i\Lambda_j = 0\,, \quad m = 1,\ldots,N. \tag{43}$$

Another relevant combination is

$$\Lambda_i - \Lambda_j = (p_{1,i} - p_{1,j})\frac{\lambda^N + \sum\limits_{k=1}^{N}\lambda^{N-k}b_{k,i}}{(\lambda - r_i)(\lambda - r_j)\sqrt{R_{2N+1}}}d\lambda\,, \tag{44}$$

where, again, the coefficients $b_{k,j}$ are given by the normalization

$$\oint_{\alpha_m} (\Lambda_i - \Lambda_j) = 0\,, \quad i \neq j\,, \tag{45}$$

that implies $b_{k,i} = a_{k,i}$. Then

$$\frac{\partial_i \Lambda_j}{\Lambda_i - \Lambda_j} = \frac{\partial_i p_{1,j}}{p_{1,i} - p_{1,j}}\,, \tag{46}$$

Recalling that $p_{1,j} = -1/4V_j$ (see (40)) we prove the identification theorem.

Q.E.D.

We also present the equivalent form of the solution (31) parametrized by the phase of the reflection coefficient from the left ρ_- (cf. (3)),

$$\rho_-(\lambda) = \int_{-\infty}^{x^-} [\lambda^{1/2} - (\lambda - u_0(x))^{1/2}]dx - \lambda^{1/2} x^- \tag{47}$$

It can be shown that the following identity holds (for details see Appendix 2)

$$\oint_{\cup\alpha_k} \{\rho_-(\lambda) + \rho_+(\lambda)\}\Lambda_j - i \oint_{\cup\beta_n} \tau(\lambda)\Lambda_j = 0\,. \tag{48}$$

Then the solution (31) takes the form

$$\oint_{a_\infty} \{x\lambda^{1/2} + 4t\lambda^{3/2}\}\Lambda_j = -\oint_{\cup\alpha_k} \rho_-(\lambda)\Lambda_j + i\oint_{\beta_E} \tau(\lambda)\Lambda_j\,, \tag{49}$$

In particular, in the case A we have especially simple representation (we note that $\cup\alpha_k = a_\infty$)

$$\oint_{a_\infty} \{x\lambda^{1/2} + 4t\lambda^{3/2} + \rho_-(\lambda)\}\Lambda_j = 0\,, \quad j = 1, \ldots, 2N+1\,. \tag{50}$$

This form of the solution coincides with the one obtained in [2].

The differential Λ_j for the first time was introduced by El [2] as a generating differential of the Whitham hierarchy and can be regarded as a nonlinear analog of the Green function for the modulation equations. Really, as Λ_j depends on a free parameter λ and satisfies identically the Tsarev equations (35), we have proved automatically that it is a fundamental solution to the Tsarev system . Expanding (32) in powers of $1/\lambda$ as $\lambda \to \infty$ we obtain the homogeneous solutions to Tsarev's equations. Those of them with the odd indices of homegeneity $n = 3, 5, 7\ldots$ are the characteristic speeds of the N-gap averaged n-th KdV in the hierarchy [12]. Therefore, Λ_j is the generating differential for the averaged KdV hierarchy. The effective formulae for the characteristic speeds of the hierarchy in terms of the generating differential are (up to a norming constant):

$$W_j^{(n)}(\mathbf{r})d\lambda = -Res_\infty \lambda^{n-3/2}\Lambda_j\,, \tag{51}$$

Corresponding solutions $r_j(x,t)$ given by the system

$$\oint_{a_\infty} \{x\lambda^{1/2} + 4t\lambda^{3/2} + c\lambda^{n-3/2}\}\Lambda_j = 0\,, \quad j = 1, \ldots, 2N+1\,. \tag{52}$$

are self-similar [12]

$$r_j(x,t) = t^\gamma l_j(\frac{x}{t^{\gamma+1}})\,, \quad \gamma = \frac{1}{n-1}\,,\ n = 3, 5, \ldots, \tag{53}$$

and solve the initial value problem

$$x = [2^n \frac{(2n-1)!!}{n!} cu^n]_{t=0}\,. \tag{54}$$

Taking $n = 3$ we arrive at the Potemin solution [28] (see also [11]) describing the universal regime of the formation of collisionless shock in the vicinity of the break point [3], [11], [16]. This solution was also obtained by Wright [29] applying the Lax-Levermore methods [6]. We also note that if

$$\rho_-(\lambda) = -\sum_{k=2}^{2N+1} c_k \lambda^{k+1/2}\,, \tag{55}$$

then the formula (50) gives the realization of the general Krichever prescription for constructing the algebraic-geometrical solutions to the Whitham equations [12].

4 Functionals of the Peierls-Fröhlich type as the Potentials for the Local Solutions to the Whitham Equations

We point out the important relationship between the differential Λ_j and the standard meromorphic differential (quasimomentum)

$$dp = \frac{\lambda^N + \sum\limits_{k=1}^{N} \lambda^{N-k} q_k}{\sqrt{R_{2N+1}}} d\lambda \tag{56}$$

normalized by

$$\oint_{\alpha_m} dp = 0\,, \quad m = 1, \ldots, N\,. \tag{57}$$

As can be easily seen,

$$\Lambda_j = \frac{\partial_j dp}{\partial_j q_1 + \frac{1}{2}} \tag{58}$$

In particular, for $N=0$:

$$\Lambda = 2\partial_r dp = \frac{d\lambda}{(\lambda - r)^{3/2}} \tag{59}$$

It is well known (see, for instance, [11]) that $\int dp$ is the generating function for the averaged Kruskal integrals Q_k:

$$P = \int dp = 2\sqrt{\lambda} + \sum_{k=0}^{\infty} \frac{2Q_k}{(2\sqrt{\lambda})^{2k+1}}\,. \tag{60}$$

Expanding (56) near infinity and comparing with (60) we find that

$$q_1 = Q_0 - \frac{1}{2}\sum_{j=1}^{2N+1} r_j\,, \tag{61}$$

where

$$Q_0 = \bar{u} = \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{L} u(x,\mathbf{r})dx \tag{62}$$

Then it follows from (58) that

$$\Lambda_j = \frac{\partial_j dp}{\partial_j \bar{u}} \tag{63}$$

provided $\partial_j \bar{u} \neq 0$. Substituting the representation (63) into the solution (31) one arrives at the potential form of the local solution to the Whitham equations

$$\partial_j F_N(\mathbf{r}; x, t) = 0\,, \quad j = 1, \ldots, 2N+1\,, \tag{64}$$

where the potential

$$F_N(\mathbf{r}; x, t) = \Phi[u_0(x); x, t, N] = \frac{1}{2\pi i}[\oint_{a_\infty} \{x\lambda^{1/2} + 4t\lambda^{3/2}\}dp + 2\int_{\cup G_k} \rho_-(\lambda)dp - 2i\int_E^1 \tau(\lambda)dp]\,, \tag{65}$$

In the case A (E=1) we observe that F_N represents the functional of the Peierls–Fröhlich type [20], [21], [22]:

$$F_N(\mathbf{r}; x, t) = \frac{1}{2\pi i}[\oint_{a_\infty} \{x\lambda^{1/2} + 4t\lambda^{3/2}\}dp + 2\int_{\cup G_k} \rho_-(\lambda)dp)]\,. \tag{66}$$

The case of the Peierls–Fröhlich type functional with (cf.(55))

$$\rho_-(\lambda) = -\sum_{k=2}^{2N+1} c_k \lambda^{k+1/2} + 2\int_{\lambda}^{\infty} \frac{g(u)}{\sqrt{u-\lambda}}du\,, \tag{67}$$

where $g(u)$ is a sufficiently small smooth function, has been studied recently by Dubrovin [19] who found that the minimizer to (66), (67) gives the solution to the Cauchy problem for the KdV-Whitham system with the monotone initial data

$$x = [\sum_{k=2}^{2N+1} \frac{(2k+1)!!}{2^{k-1}k!} c_k u^k + g(u)]_{t=0} . \tag{68}$$

One can suppose that the minimizer to the functional (65) gives rise to the solution to the Cauchy problem with hump-like initial data. We emphasize also that the solution in the form (64), (65) does not require analyticity from the initial data.

5 Global Solutions to the Whitham System

5.1 General Formulation

As it was shown in Sec.2 which gives an account of the results of [1], the steepest descent for the RH problem yeilds the global solution to the Whitham equations bypassing the procedure of integration of the modulational system itself. The obtained solution, however, is restricted by the requirements of decaying at infinity and of analyticity imposing upon the initial data. In addition, the consideration in [1] concerns only with solitonless (pure reflective) initial data. On the other hand, the Whitham equations themselves admit more general formulation of the problem cancelling these restrictions.

Our idea is to construct the global solutions to the general Cauchy problem for the Whitham equations by applying *the results* of the RH problem approach described above to a more general class of functions $\rho(\lambda)$, $\tau(\lambda)$ (and therefore to a more general class of initial data) appearing in the solution (31). More in detail, the functions $\rho(\lambda)$, $\tau(\lambda)$ can be multivalued (even infinite-valued) with different number of branches for different λ. Actually, as we will show, such type of behavior for ρ and τ corresponds to a multihump (or infinite-hump nondecaying) initial data. Also, the resulting formulation of the RH problem does not require any analyticity from the functions $\rho(\lambda)$ and $\tau(\lambda)$ which cancels requirement of analyticity for the initial data and is consistent with the hyperbolic nature of the zero-dispersion KdV limit [30].

First, we define following Dubrovin [19] the Whitham system as a sequence of the modulation systems (34) defined for $N = 1, 2, \ldots$ For $N = 0$ this coincides with the Riemann wave equation

$$\partial_t r - 6r\partial_x r = 0 , \tag{69}$$

with the initial data $r(x, 0) = u_0(x)$ are given. Solutions of the Whitham equations (34) for a given N typically exist only within certain domains of the (x, t) plane. The main problem of the theory of the Whitham equations is to glue together these solutions in order to produce a C^1-smooth multivalued function of x that also depends C^1-smoothly on the parameters $r_1, \ldots, r_{2N+1}$.

Thus, to produce the global solution to the Whitham system with the aid of the obtained local solutions (31) one should:

i) determine the right genus at every point x_0, t_0.

ii) provide C^1-smooth matching of the solutions (31) for different genera on the phase transition boundaries [11], [19].

The properties of the phase transitions can be investigated inside the local Whitham theory.

5.2 Phase Transitions

We study what happens to the solution (31) when one of the Riemann invariants r_{2j} coalesces either with r_{2j-1} or with r_{2j+1}. It follows from the solution (31) that its phase transition properties are completely determined by the properties of the differential Λ_j near the double points $r_k = r_{k+1}$ and can be investigated directly using the representation (32) for Λ_j. It is more convenient, however, to use the relationship (63) between Λ_j and the meromorphic differential dp properties of which are known well.

Near the double points, the differential dp as well as the coefficients Q_k in the decomposition (60) have the following asymptotics [19]:

1) $r_{2j+1} - r_{2j} \to 0$ (small gap)

$$f_N(r_1, \dots, r_{2N+1}) = f_{N-1}(r_1, \dots, \hat{r}_{2j}, \hat{r}_{2j+1}, \dots, r_{2N+1}) + \nu^2 f^1_{N,j}(r_1, \dots, \hat{r}_{2j}, \hat{r}_{2j+1}, \dots; \beta, \nu) + o(\nu^2)\,. \tag{70}$$

Here

$$\beta = \frac{r_{2j} + r_{2j+1}}{2}\,, \quad \nu = \frac{r_{2j+1} - r_{2j}}{2}\,. \tag{71}$$

Here and below the hat means that the correspondent coordinate is omitted.

2) $r_{2j} - r_{2j-1} \to 0$ (small band)

$$f_N(r_1, \dots, r_{2N+1}) = f_{N-1}(r_1, \dots, \hat{r}_{2j-1}, \hat{r}_{2j}, \dots, r_{2N+1}) + \delta f^2_{N,j}(r_1, \dots, \hat{r}_{2j-1}, \hat{r}_{2j}, \dots; \eta, \delta) + o(\delta)\,. \tag{72}$$

Here

$$\eta = \frac{r_{2j-1} + r_{2j}}{2}\,, \quad \delta = [\log \frac{4}{(r_{2j} - r_{2j-1})^2}]^{-1}\,. \tag{73}$$

One can see that in both cases, in the limit, the double points drop out of the function f_N and it turns into its analog for the $N-1$ genus. This fact follows from the normalization for the meromorphic differential

$$\int_{r_{2j}}^{r_{2j+1}} dp = 0\,, \qquad j = 1, \dots, N\,, \tag{74}$$

which implies that the polynomial $\lambda^N + \sum_{k=1}^{N} q_k \lambda^{N-k}$ in the numerator of dp has exactly one zero in each gap. Then, if one shrinks either gap or band, this zero inevitably coincides with the double point in the denominator.

Certainly, all the averaged Kruskal integrals $Q_k(r_1, \dots, r_{2N+1})$ (see (60)) have the same properties near the double points. Then using the representation (63) for the generating differential we arrive at the following asymptotics by differentiating (70) and (72).

1) $r_{2j+1} - r_{2j} \to 0$

$$\Lambda_k^{[N]}(r_1, \dots, r_{2N+1}; \lambda) = \Lambda_k^{[N-1]}(r_1, \dots, \hat{r}_{2j}, \hat{r}_{2j+1}, \dots, r_{2N+1}; \lambda) + O(\nu^2)\,, \qquad (75)$$

$$\text{if} \quad k \neq 2j, 2j+1\,,$$

and

$$\Lambda_k^{[N]}(r_1, \dots, r_{2N+1}; \lambda) = \Lambda_{2j+}^{[N]} + O(\nu)\,, \qquad (76)$$

$$\text{if} \quad k = 2j \quad \text{or} \quad k = 2j+1\,,$$

where $\Lambda_{2j+}^{[N]}$ is the limiting value of the differential $\Lambda_k^{[N]}$ which follows from (32), (33) when one pinches the j-th gap :

$$\Lambda_{2j+}^{[N]} \equiv \Lambda_k^{[N]}(r_1, \dots, r_{2j-1}, r_{2j}, r_{2j}, r_{2j+2}, \dots, r_{2N+1}; \lambda) = \frac{\lambda^N + \dots}{(\lambda - r_{2j})^2 \sqrt{R'_{2N-1}}} d\lambda\,, \qquad (77)$$

$$R'_{2N-1} = (\lambda - r_1)(\lambda - r_2) \dots (\lambda - r_{2j-1})(\lambda - r_{2j+2}) \dots (\lambda - r_{2N+1}),$$

and $\Lambda_{2j+}^{[N]}$ has a double pole at $\lambda = r_{2j}$.

2) $r_{2j} - r_{2j-1} \to 0$

$$\Lambda_k^{[N]}(r_1, \dots, r_{2N+1}; \lambda) = \Lambda_k^{[N-1]}(r_1, \dots, \hat{r}_{2j-1}, \hat{r}_{2j}, \dots, r_{2N+1}; \lambda) + O(\delta)\,, \qquad (78)$$

$$\text{if} \quad k \neq 2j-1, 2j\,,$$

and

$$\Lambda_k^{[N]}(r_1, \dots, r_{2N+1}; \lambda) = \Lambda_{2j-}^{[N]} + O(\nu/\delta)\,, \qquad (79)$$

$$\text{if} \quad k = 2j-1 \quad \text{or} \quad k = 2j\,,$$

where the $\Lambda_{2j-}^{[N]}$ is the limiting value of the differential $\Lambda_k^{[N]}$ when one pinches the j-th band :

$$\Lambda_{2j-}^{[N]} \equiv \Lambda_k^{[N]}(r_1, \dots, r_{2j-2}, r_{2j}, r_{2j}, r_{2j+1}, \dots, r_{2N+1}; \lambda) = \frac{\lambda^{N-1} + \dots}{(\lambda - r_{2j}) \sqrt{R'_{2N-1}}} d\lambda\,, \qquad (80)$$

$$R'_{2N-1} = (\lambda - r_1)(\lambda - r_2) \dots (\lambda - r_{2j-2})(\lambda - r_{2j+1}) \dots (\lambda - r_{2N+1}),$$

and $\Lambda_{2j-}^{[N]}$ has a single pole at $\lambda = r_{2j}$ (cancellation of one pole occurs due to the zero in the vanishing j-th band).

One can see the substantial difference in the limiting behavior of the differential $\Lambda_{2j}^{[N]}$ depending whether one shrinks the j-th gap (77) or the j-th band (80).

The asymptotics (75) and (78) provide natural C^1 - smooth matching of the solution (31) for different genera on the phase transition boundaries. The limiting values in (76) and (79) determine the motion of those boundaries. Namely, the boundaries $x_{j\pm}$ separating N-phase and $N-1$-phase regions correspond to closing either the j-th gap (+) (linear wave degeneration) or the j-th band ($-$) (soliton degeneration) and obey the ODE's :

$$\frac{dx_{j\pm}}{dt} = -4 Res_{\infty} \lambda^{3/2} \Lambda_{2j\pm}^{[N]} \qquad (81)$$

One should also check what happens to the solutions (31) under the phase transition $(N=1) \to (N=0)$. As $N=0$ we have $\Lambda_j^{[0]} = (\lambda - r)^{-3/2} d\lambda$ (see (59)) and, as a result, we arrive at two different equations corresponding to the cases A and B respectively. In the case A we have

$$\text{A:} \qquad x - \frac{2t}{\pi i} \oint\limits_{a_\infty} \frac{\lambda^{3/2}}{(\lambda - r)^{3/2}} d\lambda = \frac{2}{\pi} \frac{d}{dr} \int\limits_0^r \frac{\rho_-(\lambda)}{\sqrt{r-\lambda}} d\lambda \,, \tag{82}$$

which is the solution of the Riemann wave equation (69) with the initial data in the form of the increasing part of the hump $u_0(x)$:

$$x + 6rt = x^-(r) \,. \tag{83}$$

For the case B we have an analogous result for the decreasing part of the initial hump u_0 :

$$\text{B:} \qquad x + 6rt = x^-(r) - \frac{2}{\pi} \frac{d}{dr} \int\limits_r^1 \frac{\tau(\lambda)}{\sqrt{\lambda - r}} d\lambda = x^+(r) \,. \tag{84}$$

It can be easily seen also that the decompositions of $\Lambda_j^{[1]}$ near $\Lambda_j^{[0]}$ have the form (75), (78) that provides C^1 - smoothness of the transition $N=1 \to N=0$. This transition was investigated in detail by Avilov, Krichever and Novikov [31] (see also [11], [19]).

It is clear now that the cases A and B in the solution (31) describe different parts of the global solution of the Whitham equations corresponding to the monotonic branches of the initial profile $u_0(x)$. We also note that formulas (82), (84) give the solution to the Riemann equation in terms of the KdV scattering data and were obtained in this form by Geogjaev in [32].

5.3 Inequalities: Determination of Genus

As it has been said before, the necessary part of constructing the solution to the Cauchy problem is determination of the genus of the Riemann surface at every point x, t. In fact, this problem cannot be resolved locally in the frame of the obtained solution (31) for the Whitham system. This point of view has not been established clearly in the literature so we will describe it more in detail.

The genus (the number of oscillating phases) N is the "external" characteristics with respect to the local solutions to the Whitham system. To illustrate that more clearly, we present a simple example of ambiguity in the local determination of genus.

In the Figure 1, there is a typical solution of one-phase averaged Whitham equations for the initial data with the only break point (for example, the cubic breaking in the Gurevich – Pitaevskii problem [3]) depicted by a solid line in the region (x_-, x_+) while the three-valued solution to the Riemann wave equation (69) is depicted by a broken line in the region (x_1, x_2), $x_1 < x_- < x_2 < x_+$. Outside of those intervals both solutions coincide.

One can see that in the vicinity of any point of an open interval (x_2, x_+) both cases $g=0$ and $g=1$ can be applied without any violations of the existence for the local solution.

The missing part of the information which enables to distinguish the unique solution in this interval is the break time t_{crit} (if $t > t_{crit}$ then $N = 1$ otherwise $N = 0$) but this information is relevant to the global properties of the solution for case $N = 1$. In other words, generically to make the right decision about the genus at $t = t_0$ one should follow the solution all time until t_0 and change genus after passing the critical points . The rigorous global predictions of the genus have been made so far only in a few simple cases.

In the case of analytic initial data with the only break point there are results by Tian [14], [15] that the solution beyond the break time globally belongs to $N = 1$.

Another result in this area is due to Grava [18] and it reads that the maximal genus in the Cauchy problem with polynomial initial data does not exeed the degree of the polynomial and $N \to 1$ asymptotically as $t \to \infty$ (the latter result also can be found in [2]).

But even in the simplest case of the polynomial data the determination of genus at a fixed point x_0, t_0 requires knowledge of the behaviour of the solution in all earlier times.

The RH problem approach makes it possible to determine the right genus of the problem locally with the aid of the inequalities (8) – (12) for the complex phase function $g(\lambda)$. These original inequalities were derived for the decaying solitonless initial data in [1], [25]. It is clear, however, that if one assumes the finite speed of propagation (hyperbolicity) in the zero dispersion limit (which has been rigorously proved for the case of decaying initial potential [6], [7], [30]) then one can consider an arbitrary initial profile and regard these inequalities as a complementary part to the general local solution (31) for finite t. Really, given the functions $\rho_\pm(\lambda)$ and $\tau(\lambda)$ as the simple integral (Abel) transforms (3),(47), (4) of arbitrary initial data one can construct the function $g(\lambda)$ and its derivative $g'(\lambda)$ and then, by trial and error, to determine the genus with the aid of inequalities. Also, one should add the relations (9) and (10) distinguishing the cases A and B for the problem with nonmonotone initial data. Certainly, for the monotonically increasing data we always are in the case A.

Example: One-Hump Problem

Following [1] we present now an example of a systematic procedure for obtaining $N = N(x,t)$ for all x at any given t. The result of this procedure is to construct for each t the *separatrix* $F(t) = \{(x,r) : r = r(x,t), -\infty < x < \infty, r = r_1, \ldots, r_{2N+1}\}$ for the case of one-hump initial data with finite number of break points (see Figures 2a and 2b).

For times t less than some critical value t_{crit} one takes $N = N(x,t) = 0$ and solves (69) for r; the time t_{crit} corresponds precisely to the time at which (1) with $\epsilon = 0$ breaks down. It turns out that the associated function g constructed as above indeed satisfies the auxilliary inequalities (6), (8), (9), (10) and hence g is the desired solution. Thus, for $t < t_{crit}$, we obtain a separatrix of the shape of Figure 2a; moreover, we find that $F(t) = (x,r) : r = u(x,t)$, where $u(x,t)$ is the solution of (1) with $\epsilon = 0$. For $x > x_0(t)$, we are in case B and for $x < x_0(t)$, we are in case A. When $t > t_{crit}$, one again sets $N = N(x,t) = 0$ for $x >> 1$, computes r from (69) and verifies once again that the side conditions are satisfied in case B. However, as we move x to the left, we find that at least one of the inequalities (6), (8), (9), (10) breaks down. In Figure 2b below the first inequality in (6), $-\tau < (g_+ - g_-)/2i$, breaks down at $x = x_1$ and for $x < x_1$ the interval G_0 breaks up into two intervals G_0 and G_1. For such x one solves the system (31) for one of the cases A or B to get r_1, r_2, and r_3 and verifies that indeed the associated N satisfies the above inequalities. In the scenario of

Figure 2b, as we move x further to the left, the same inequality for τ again breaks down at some $x = x_2(t)$, for some $r \in G_0(x_2(t), t)$. Again G_0 splits up into two intervals yielding a total of three intervals G_0, G_1 and G_2, etc. As we continue to move towards x_3, the interval G_1 shrinks to a point and eventually disappears. For x between x_4 and x_3 we are again in the two interval case and for $x < x_4$ we return to the one interval case $N = 0$. For $x > x_0$, we are always in case B and for $x < x_0$ we are always in case A. The scenario of figure 2b is representative of the generic case in which any finite number of intervals may be open at some x. For more complicated, indeed pathological initial data infinitely many folds may appear.

6 Zero-Dispersion KdV with General Initial Data

The basic assumption which is made in the following consideration is the hyperbolicity of the zero dispersion limit for the KdV with an arbitrary initial potential. By applying this assumption one can extend the formulation of the RH problem made in Sec.2 for the case of the solitonless alanytic initial perturbation to the case of an arbitrary initial data and to construct the solution to the corresponding Cauchy problem for the KdV-Whitham system.

It is clear that due to the (assumed) hyperbolicity of the problem only a finite number of 'humps' in the initial perturbation will be involved into nonlinear interaction at any finite time at any particular point. Therefore, the solution to the problem for the Whitham equations with an arbitrary (nondecaying) initial data can be obtained by solving the "multi-hump" (decaying) problem in the zero dispersion limit and then by tending the number of humps to infinity. Certainly, the solution obtained in this way will be valid during limited time interval (presumably for $t \ll 1/\epsilon$) until the semiclassical spectrum (Riemann invariants of the Whitham equations) begins to compete with the fine spectrum of the initial potential and the problem loses its hyperbolic character.

The main technical obstacle to the direct extension of the formulation (5) – (12) to the case of a multi-hump initial data is the multivaluedness of the functions $\rho_+(\lambda)$ and $\tau(\lambda)$ (3), (4) appearing in the RH problem. To manage this difficulty it is convenient to first reformulate the RH problem for one-hump initial data introducing the finite reference point instead of the reference point at infinity (scattering from the right).

6.1 Reformulation of the One-Hump Problem for the Finite Reference Point

We start with the function

$$g(\lambda, x, 0) = \int_x^\infty (\lambda^{1/2} - (\lambda - u_0(x'))^{1/2}) dx' \,, \tag{85}$$

which, as can be checked directly, solves the RH problem (5) – (12) for $t = 0, \quad N = 0$.

We introduce a new function

$$H(\lambda,x,x_0)=\int\limits_x^{x_0}(\lambda^{1/2}-(\lambda-u_0(x'))^{1/2})dx'\,, \tag{86}$$

where x_0 is an arbitrary fixed reference point. Without loss of generality we put x_0 at the maximum of $u_0(x)$ (see Fig.2a).

One can see that

$$g_\pm(\lambda,x,0)=H_\pm-\lambda^{1/2}x_0+\rho_+\mp i\tau_+\,, \tag{87}$$

where

$$\tau_+(\lambda)=\int\limits_{x_0}^{x^+}(u_0(x)-\lambda)^{1/2}dx\,. \tag{88}$$

Then we have the relationships (at $t=0$)

$$\frac{g_++g_-}{2}=\frac{H_++H_-}{2}-\lambda^{1/2}x_0+\rho_+\,, \tag{89}$$

$$\frac{g_+-g_-}{2i}=\frac{H_+-H_-}{2i}-\tau_+\,. \tag{90}$$

We also introduce

$$\tau_-(\lambda)=\tau_+(\lambda)-\tau(\lambda)=\int\limits_{x_0}^{x^-}(u_0(x)-\lambda)^{1/2}dx\,. \tag{91}$$

Then the function $H(\lambda,x,x_0)$ is the solution of the RH problem which follows directly from (5) – (12) :

In the region $x<x^-$ (see Fig.2a) which lies on the left from the hump and contributes to the spectral band B_1 the correspondent relations follow from (9) (case A) taking into account (89), (90):

$$\frac{H_+-H_-}{2i}=\tau_-\,,\qquad \frac{H'_++H'_-}{2}<-\alpha'_0\,, \tag{92}$$

where

$$\alpha_0(\lambda,x)=\lambda^{1/2}(x-x_0)\,. \tag{93}$$

Analogously, for the region $x^-<x<x^+$ lying under the hump and contributing to the gap G_0 we have from (5), (6):

$$\frac{H'_++H'_-}{2}=-\alpha'_0\,,\qquad \tau_-<\frac{H_+-H_-}{2i}<\tau_+\,. \tag{94}$$

And finally, in the region $x>x^+$ which lies on the right from the hump and again contributes to the band B_1 (but it is the case B now) in the original RH problem (see (10)) we have

$$\frac{H_+-H_-}{2i}=\tau_+\,,\qquad \frac{H'_++H'_-}{2}>-\alpha'_0\,. \tag{95}$$

Also we have outside the interval $(0,1)$ from (11), (12)

$$H_+ + H_- = 0 \qquad \text{if} \qquad \lambda < 0, \tag{96}$$

$$H_+ - H_- = 0 \qquad \text{if} \qquad \lambda > 1\,. \tag{97}$$

Due to the finite speed of propagation in the zero dispersion KdV with decaying initial data the topology of the picture (Fig.2a) does not change under the t-evolution (see Fig 2b). Finitely many folds which appear at some finite t determine the band-gap structure at any point x so that the region $x < x^-$ on the $(u,\ x)$ - plane of the initial data will provide the bands $B_n,\ (n = 1, \ldots, N+1)$ (B_{N+1} corresponds here to the case A), the region $x^- < x < x^+$ will provide the gaps $G_k,\ (k = 0, \ldots, N)$, and the region $x > x^+$ always will correspond to the band B_{N+1} in the case B. Then one can formulate the time-dependent RH problem for the finite t by the substitution reflecting the linear temporal evolution of the phase $\alpha(\lambda)$ (2):

$$\alpha_0(\lambda, x) \to \alpha_0(\lambda, x, t) = \lambda^{1/2}(x - x_0) + 4\lambda^{3/2} t \tag{98}$$

in the formulas (92) – (97).

Thus, we have effectively removed the infinite reference point from the formulation of the RH problem corresponding to the evolution of the one-hump perturbation.

6.2 Evolution of an Arbitrary Profile

The strategy of investigation of the evolution for an arbitrary initial perturbation $u_0(x)$ is essentially the same as in the previous subsection: we formulate the RH problem for the function $H(\lambda, x, x_0)$ (86) at $t = 0$ starting with the fixed reference point x_0 and then by *assuming* the finite speed of propagation (hyperbolicity of the zero-dispersion KdV) arrive at the time-dependent H-function by the substitution (98) for the phase α_0 . As was mentioned, the situation, however, is complicated by the multivaluedness of the functions $\rho(\lambda)$ and $\tau(\lambda)$ appearing in such a problem.

We begin with the multihump initial potential and split up the strip $0 < u < 1$ in the u, x - plane of the initial data into two type of domains (see Fig.3a):

τ - domains correspond to humps in the intitial perturbation and contribute to the gaps in the semiclassical spectrum.

ρ - domains correspond to wells in the initial perturbation and contribute to the bands in the semiclassical spetrum.

We fix the reference point x_0 at some maximum of the function $u_0(x)$ and enumerate the monotonic parts of $u_0(x)$ for $x > x_0$ labelling them by the roots of the equation $\lambda = u_0(x)$ (to avoid unnecessary notation complexities we put the number of humps on the right from the point x_0 equal to the number of humps on the left):

$$x^-_{-M} < x^+_{-M+1} < \ldots < x^+_{-1} < x^-_{-1} \le x_0 \le x^+_1 < x^-_1 < \ldots < x^-_{M-1} < x^+_M\,, \tag{99}$$

$$M \in \mathbf{N}\,, \quad \frac{dx^-_m}{d\lambda} > 0\,, \qquad \frac{dx^+_m}{d\lambda} < 0\,.$$

We denote the domains of definition for each $x^{\pm}_m(\lambda)$ as $D^{\pm}_m$. Then for each $\lambda = \lambda^*$ one can define two subsequences: $\{x^-_{m_j}(\lambda^*)\}$ and $\{x^+_{m_j}(\lambda^*)\}$, ($m_j = \pm 1, \pm 2, \dots, \pm M_j$, $j = j(\lambda) = 1, \dots, 2M-1$, $x^+_{k_j} > x^-_{k_j-1}$) such that only those $x^{\pm}_k$ participate in $\{x^{\pm}_{m_j}\}$ for which $\lambda^* \in D^{\pm}_k$ (Fig3.b).

For each $j = j(\lambda)$ one can then define the sequences $\{\rho_{m_j}\}$, $\{\tau_{m_j}\}$:

$$\rho_{m_j}(\lambda) = Re \int\limits_{x_0}^{x^-_{m_j}} (\lambda - u_0(x'))^{1/2} dx' \,, \qquad \tau_{n_j}(\lambda) = Im \int\limits_{x_0}^{x^+_{n_j}} (\lambda - u_0(x'))^{1/2} dx' \tag{100}$$

Then the function $H(\lambda, x, x_0, 0)$ defined by (86) and corresponding now to a multi-hump analytic initial data $u_0(x)$ is related to the original function $g(\lambda, x, 0)$ by (cf. (87))

$$\text{if} \;\; x^-_{m_j-1} < x < x^-_{m_j} \,, \;\; m_j > 0 : \qquad g_{\pm} = H_{\pm} - \lambda^{1/2} x_0 + \rho_+ - \rho_{m_j-1} \mp i\tau_{m_j} \,, \tag{101}$$

where $\rho_+(\lambda)$ is redefined for the multi-hump perturbation in the following way:

$$\rho_+(\lambda) = x^+_{m_j} \lambda^{1/2} + \int\limits_{x^+_{m_j}}^{\infty} [\lambda^{1/2} - (\lambda - u_0(x'))^{1/2}] dx' \,, \tag{102}$$

Here one should put $\rho_0 \equiv 0$ and $x^{\pm}_0 \equiv x_0$. If $m_j < 0$ then the relationship (101) is valid within the interval $x^+_{m_j} < x < x^+_{m_j+1}$.

Then, at $t = 0$ we have instead of (89), (90) within the mentioned intervals covering the entire axis:

$$\frac{g_+ + g_-}{2} = \frac{H_+ + H_-}{2} - \lambda^{1/2} x_0 + \rho_+ - \rho_{m_j-1} \,, \tag{103}$$

$$\frac{g_+ - g_-}{2i} = \frac{H_+ - H_-}{2i} - \tau_{m_j} \,. \tag{104}$$

Now we can formulate the RH problem which is satisfied by the function H. We substitute (103), (104) into the original RH problem (5) – (12) taking into account new definition of $\rho_+(\lambda)$ (102):

In each τ - domain ($x^-_{m_j-1} < x < x^+_{m_j}$ for $m_j > 0$ and $x^-_{m_j} < x < x^+_{m_j+1}$ for $m_j < 0$) one has from (5), (6)

$$\frac{H'_+ + H'_-}{2} = -\alpha'_0 + \rho'_{m_j-1} \,, \qquad \tau_{m_j-1} < \frac{H_+ - H_-}{2i} < \tau_{m_j} \,. \tag{105}$$

In each ρ - domain $x^+_{m_j} < x < x^-_{m_j}$ we have from (7), (8), (10)

$$\frac{H_+ - H_-}{2i} = \tau_{m_j} \,, \qquad -\alpha'_0 + \rho'_{m_j-1} < \frac{H'_+ + H'_-}{2} < -\alpha'_0 + \rho'_{m_j} \,. \tag{106}$$

We note that inequalities (8), (9), (10) have converted into one double inequality. One should also put $\tau_0 \equiv 0$.

Outside the interval $(0,1)$ one has analogously to (96), (97):

$$H_+ + H_- = 0 \qquad \text{if} \qquad \lambda < 0, \tag{107}$$

$$H_+ - H_- = 0 \qquad \text{if} \qquad \lambda > 1\,. \tag{108}$$

As well as in the previous subsection, the time-dependent RH problem is obtained from (105) – (108) by adding the term $4\lambda^{3/2}t$ to $\alpha_0(\lambda, x)$ (see (98)). However, to retain validity of the formulated RH problem after this substitution one should *assume* hyperbolicity of the zero-dispersion limit for the KdV with general initial data. In this case the toplogy of the separatrix would not change under the evolution.

Recalling that τ - regions contribute to the gaps $G_k\,, (k = 0, \ldots, N)$ and ρ - regions contribute to the bands $B_n\,, n = 1, \ldots, N+1$, one immediately obtains the function $H(\lambda, x, x_0, t)$ and its derivative (cf. (19), (15)). For the sake of simplicity we write down the result assuming that the initial data are such that each ρ - domain contributes to only one band (analogously, each τ – domain contributes to only one gap). Consideration of more general data would only complicate notations adding nothing to substance.

$$H'(\lambda, x, x_0, t) = \sqrt{R_{2N+1}(\lambda)} \sum_{m=0}^{N} \left(\int_{G_m} \frac{2{\rho_m^*}'(\mu) - 2\alpha_0'(\mu, x, t)}{\sqrt{R_{2N+1}^+}(\mu - \lambda)} \frac{d\mu}{2\pi i} + \int_{B_{m+1}} \frac{-2i{\tau_m^*}'(\mu)}{\sqrt{R_{2N+1}^+}(\mu - \lambda)} \frac{d\mu}{2\pi i} \right), \tag{109}$$

$$H(\lambda, x, x_0, t) = \sqrt{R_{2N+1}(\lambda)} \sum_{m=0}^{N} \left(\int_{G_m} \frac{2\rho_m^*(\mu) - 2\alpha_0(\mu, x, t) - \Omega_m}{\sqrt{R_{2N+1}^+}(\mu - \lambda)} \frac{d\mu}{2\pi i} + \int_{B_{m+1}} \frac{-2i\tau_m^*(\mu)}{\sqrt{R_{2N+1}^+}(\mu - \lambda)} \frac{d\mu}{2\pi i} \right), \tag{110}$$

where

$$\Omega_m = -2(x - x_0) \oint_{a_\infty} \lambda^{1/2}\psi_j - 8t \oint_{a_\infty} \lambda^{3/2}\psi_j + 4\sum_{m=0}^{N} \Big(\int_{G_m} \rho_m^* \psi_j - i \int_{B_{m+1}} \tau_m^* \psi_j \Big)\,, \tag{111}$$

$$\Omega_0 = 0, \qquad \rho_0^* = 0$$

. Here

$$\{\rho_m^*(\lambda)\} \subset \{\rho_{k_j}(\lambda)\}\,, \qquad \{\tau_m^*(\lambda)\} \subset \{\tau_{k_j}(\lambda)\}\,, \qquad m = 0, \ldots, N\,. \tag{112}$$

The algebraic equations determining dependence of the band-gap structure on x and t are obtained from the condition (30) $\partial_j \Omega_i = 0$ and can be eventually represented in a potential form (64) with the generalized Peierls - Fröhlich type functional as a potential (see Sec.3, 4)

$$\partial_j F_N(\mathbf{r}; x, t) = 0\,, \quad j = 1, \ldots, 2N+1\,, \tag{113}$$

$$F_N(\mathbf{r}; x, t) = \frac{1}{2\pi i}\Big[\oint_{a_\infty} \{(x - x_0)\lambda^{1/2} + 4t\lambda^{3/2}\} dp - 2\sum_{m=0}^{N} \Big(\int_{G_m} \rho_m^* dp - i \int_{B_{m+1}} \tau_m^* dp \Big) \Big]. \tag{114}$$

The system (113), (114) represents the general local solution to the Whitham equations. Due to the properties of the meromorphic differential dp discussed in Sec.5.2 this solution provides C^1 - smooth matching on the phase transition boundaries where genus N of the hyperelliptic surface changes. The right choice of the local genus N and the subsets $\{\tau_m^*\}$, $\{\rho_m^*\}$ (112) is verifed by check of the fulfilment of the inequalities (105) and (106) for the function $H(\lambda, x_0, x, t)$.

As an example of an effective choice of the subsets $\{\tau_m^*\}$, $\{\rho_m^*\}$ one can indicate the case of periodic initial data $u_0(x)$ where $\tau_m^* = (m+1/2) Im \int_{x_0}^{x_0+T} (\lambda - u_0(x'))^{1/2} dx'$, $\rho_m^* = mRe \int_{x_0}^{x_0+T} (\lambda - u_0(x'))^{1/2} dx'$, T is the period, $m \in \mathbf{N}$

Thus, we have bypassed the inverse scattering transform for the zero - dispersion KdV with general initial data by assuming hyperbolicity of the zero dispersion limit. One can see that the resulting construction requires neither decaying nor analyticity for the initial data.

Aknowledgements

The research described in this paper was made possible by the financial support of the Civilian Research and Development Foundation for the Former Soviet Union and USA (CRDF) under the grant # RM1 - 145.

G.A.E and A.L.K thank also the Russian Foundation for Basic Research (RFBR) for the partial financial support under the grant # 00 - 01 - 00210.

Appendix 1: Proof of the Consistency Lemma

Consider the function

$$\Phi_i^{(j)} = \frac{\partial_i \Omega_j}{\partial_i k_j}\,. \tag{A1.1}$$

We rewrite the expression (20) in the form containing contour integrals. For analytic functions $\rho_+(\lambda)$ and $\tau(\lambda)$ we have

$$\Omega_j = \oint_{a_\infty} \{-2x\lambda^{1/2} - 8t\lambda^{3/2}\}\psi_j + \oint_{\cup\alpha_k} 2\rho_+(\lambda)\psi_j - 2i \oint_{\cup\beta_n \setminus \beta_E} \tau(\lambda)\psi_j\,. \tag{A1.2}$$

Here we have used the fact that the functions $\rho_+(\lambda)$ (3) and $\tau(\lambda)$ (4) have their own different Riemann surfaces with the branch points:

$$0,\,\infty \qquad \text{for} \qquad \rho_+(\lambda)$$

$$1,\,\infty \qquad \text{for} \qquad \tau(\lambda)$$

Then we obtain from (A1.1)

$$\Phi_i^{(j)} = -2\oint_{a_\infty} \{x\lambda^{1/2} + 4t\lambda^{3/2}\}\frac{\partial_i \psi_j}{\partial_i k_j} + 2\oint_{\cup\alpha_k} \rho_+(\lambda)\frac{\partial_i \psi_j}{\partial_i k_j} - 2i \oint_{\cup\beta_n \setminus \beta_E} \tau(\lambda)\frac{\partial_i \psi_j}{\partial_i k_j}\,. \tag{A1.3}$$

We consider the differential

$$\Lambda_i^{(j)} = \frac{\partial_i \psi_j}{\partial_i k_j}\,. \tag{A1.4}$$

Substituting (22) into (A1.4) we immediately obtain

$$\Lambda_i^{(j)} = \frac{\lambda^N + \sum\limits_{k=1}^{N} \lambda^{N-k} p_{k,i}^{(j)}}{(\lambda - r_i)\sqrt{R_{2N+1}}} d\lambda \tag{A1.5}$$

We integrate $\Lambda_i^{(j)}$ over the α-cycles. Then, taking into account (23), (A1.4) we get

$$\oint\limits_{\alpha_m} \Lambda_i^{(j)} = 0, \tag{A1.6}$$

$$m, j = 1, \ldots, N\,,\ \ i = 1, \ldots, 2N+1$$

The normalization (A1.6) uniquely defines the coefficients $p_{k,i}$ independently on j, i.e.

$$\Lambda_i^{(1)} = \Lambda_i^{(2)} = \ldots = \Lambda_i^{(N)} \equiv \Lambda_i\,. \tag{A1.7}$$

Therefore, the differential (A1.4) does not depend on the index j, and, according to (A1.3), the function $\Phi_i^{(j)}$ does not depend on j as well. That means that the system (30) is cosistent and takes the form

$$\oint\limits_{a_\infty} \{x\lambda^{1/2} + 4t\lambda^{3/2}\}\Lambda_j = \oint\limits_{\cup\alpha_k} \rho_+(\lambda)\Lambda_j - i \oint\limits_{\cup\beta_n \backslash \beta_E} \tau(\lambda)\Lambda_j\,, \tag{A1.8}$$

provided $\partial_i k_j \neq 0$.

Q.E.D.

Appendix 2: Derivation of the Identity (48)

We introduce the analytic function

$$f(\lambda) = \int\limits_{-\infty}^{\infty} [\lambda^{1/2} - (\lambda - u_0(x'))^{1/2}]dx' = \rho_-(\lambda) + \rho_+(\lambda) - i\tau(\lambda)\,. \tag{A2.1}$$

Then one can observe that the following identity holds

$$\oint\limits_{a_\infty} f(\lambda)\Lambda_j = 0\,. \tag{A2.2}$$

Really, $f(\lambda) \sim \lambda^{-1/2}$ and $\Lambda_j \sim \lambda^{-3/2} d\lambda$ as $\lambda \to \infty$. Considering (A2.2) as the integral over the contour surrounding the interval $(0,1)$ we arrive at the identity for analytic $\rho_-(\lambda)$, $\rho_+(\lambda)$ and $\tau(\lambda)$.

$$\oint\limits_{\cup\alpha_k} \{\rho_-(\lambda) + \rho_+(\lambda)\}\Lambda_j - i \oint\limits_{\cup\beta_n} \tau(\lambda)\Lambda_j = 0\,. \tag{A2.3}$$

References

[1] P.Deift, S.Venakides, and X.Zhou, *New results in small dispersion KdV by an extension of the steepest descent method for Riemann-Hilbert problems*, International Math. Research. Journ., No 4 (1997), 285-299.

[2] G.A.El, *Generating function for the Whitham-KdV hierarchy and effective solution of the Cauchy problem*, Phys.Lett.A **222** (1996), 393-399.

[3] A.V.Gurevich and L.P.Pitaevsky, *Nonstationary structure of a collisionless shock wave*, Sov.Phys.JETP **38** (1974), 291-297.

[4] G.B.Whitham, *Non-linear dispersive waves*, Proc. Roy. Soc. **A283** (1965), 238-291.

[5] H. Flaschka, G. Forest, D.W. McLaughlin, *Multiphase averaging and the inverse spectral solutions of the Korteweg – de Vries equation*, Comm. Pure Appl. Math. **33** (1979), 739-784 .

[6] P.D.Lax and C.D.Levermore, *The small dispersion limit of the Korteweg – de Vries equation I, II, III*, Comm. Pure Appl. Math. **36** (1983) 253-290, 571-593, 809-829.

[7] S.Venakides, *The zero dispersion limit of the Korteweg – de Vries equation for initial potentials with nontrivial reflection coefficient*, Comm. Pure Appl. Math. **38** (1985), 125-155 .

[8] S. Venakides, *The generation of modulated wavetrains in the solutionof the Korteweg – de Vries equation*, Comm. Pure Appl. Math. **38** (1985), 883-909.

[9] S. Venakides, *The zero dispersion limit of the Korteweg – de Vries equation with periodic initial data*, AMS Transaction **301** (1987), 189-226.

[10] S.P.Tsarev, *On Poisson brackets and one-dimensional systems of hydrodynamic type*, Soviet Math. Dokl. **31** (1985), 488-491 .

[11] B.A.Dubrovin and S.P.Novikov, *Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory*, Russian Math. Surveys **44** (1989), 35-124 .

[12] I.M. Krichever, *The method of averaging for two-dimensional "integrable" equations*, Funct. Anal. Appl. **22** (1988), 200-213.

[13] I.M.Krichever, *The τ-function of the universal Whitham hierarchy, matrix models and topological field theories*, Comm.Pure Appl.Math., **47** (1994), 437-475.

[14] F.R.Tian, *Oscillations of the zero dispersion limit of the Korteweg – de Vries equation*, Comm. Pure Appl. Math. **46** (1993), 1093-1129.

[15] F.R.Tian, *The initial value problem for the Whitham modulation equations*, Comm. Math. Phys. **166** (1994), 79-115 .

[16] A.V.Gurevich, A.L.Krylov and G.A.El, *Riemann wave breaking in dispersive hydrodynamics*, JETP Lett. **54** (1991), 102-107 , *Evolution of a Riemann wave in dispersive hydrodynamics*, Sov. Phys. JETP **74** (1992), 957-962.

[17] A.V.Gurevich, A.L.Krylov, N.G.Mazur and G.A.El, *Evolution of a localized perturbation in Korteweg – de Vries hydrodynamics*, Sov. Phys. Dokl. **37** (1992), 198-200 , A.L.Krylov, V.V.Khodorovskii, G.A.El, *Evolution of a nonmonotonic perturbation in Korteweg – de Vries hydrodynamics*, JETP Lett. **56** (1992), 323-327 .

[18] T.Grava, *On the Cauchy Problem for the Whitham Equations*, Ph.D Thesis, SISSA (1998).

[19] B.Dubrovin, *Functionals of the Peierls – Fröhlich type and the variational principle for the Whitham equations*, Amer.Math.Soc.Transl.(2) **179** (1997), 35-44.

[20] E.D.Belokolos, *Peierls – Fröhlich problem and potentials with finite number of gaps I, II*, Theor.Math.Phys, **45** (1980), 1022-1026, **48** (1081), 604-610.

[21] I.M. Krichever, *The Peierls model*, Func. Anal. Appl. **16** (1982), 248-263.

[22] S.A. Brazovskii, I.E.Dzyaloshinskii, I.M.Krichever, *Discrete Peierls – Fröhlich models with exact solutions*, Sov.Phys.JETP **56** (1982), 212-225.

[23] A.B.Shabat, *One dimensional perturbations of a differential operator and the inverse scattering problem*, Problems in Mechanics and Mathematical Physics, Nauka, Moscow, 1976.

[24] B.Beals, P.Deift, and C.Tomei, *Scattering and inverse scattering for first order systems*, AMS, Math.Surveys and Monographs No.28 (1988).

[25] S.Venakides, *The small dispersion KdV equation with decaying initial data*, Proc.Symp. Appl. Math **54** (1998), 369-392.

[26] V.R.Kudashev, *"Waves-number conservation" and succession of symmetries during a Whitham averaging*, JETP Lett. **54** (1991), 175-178.

[27] P.D.Lax, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, Conference Board of the Mathematical Sciences, Regional Conference Series in Applied Mathematics, SIAM, Philadelphia (1973).

[28] G.V.Potemin, *Algebro-geometric construction of self-similar solutions of the Whitham equations*, Russian Math. Surveys, **43** (1988), 252-253.

[29] O.Wright, *Korteweg – de Vries zero-dispersion limit: through first breaking for cubic-like analytic initial data*, Comm.Pure Appl.Math., **46** (1993), 421-438.

[30] C.D.Levermore, *The hyperbolic nature of the zero dispersion KdV limit*, Comm. Partial Differential Equations **13** (1988), 495-514.

[31] V.V.Avilov, S.P.Novikov, *Evolution of Whitham zone in KdV theory*, Sov.Phys.Dokl. **32** (1987), 366-368; V.V.Avilov, I.M. Krichever, S.P.Novikov, *Evolution of the Whitham zone in the Korteweg – de Vries theory*, 564-566.

[32] V. V. Geogjaev, *On the quasiclassical approach in the inverse scattering transform method*, Dokl. Akad. Nauk SSSR **284** (1985), 1093-1097 (in Russian).

Figure Captions

- 1. Ambiguity in local determination of genus. Solid line: solution of the Whitham equations ($N = 1$), broken line: solution of the Hopf equation ($N = 0$).

- 2. Separatrix $\{F(t) = \{(x, r) : r(x, t) = r_1, \ldots, r_{2N+1}\}\}$, one-hump case.
  a) $F(0) = \{(x, r) : \ r = u_0(x)\}$ b) $F(t), \quad t > t_{crit}$

- 3. Multi-hump initial data. a) ρ- and τ- domains. b) Sequence $\{x^{\pm}_{k_j}\}$.

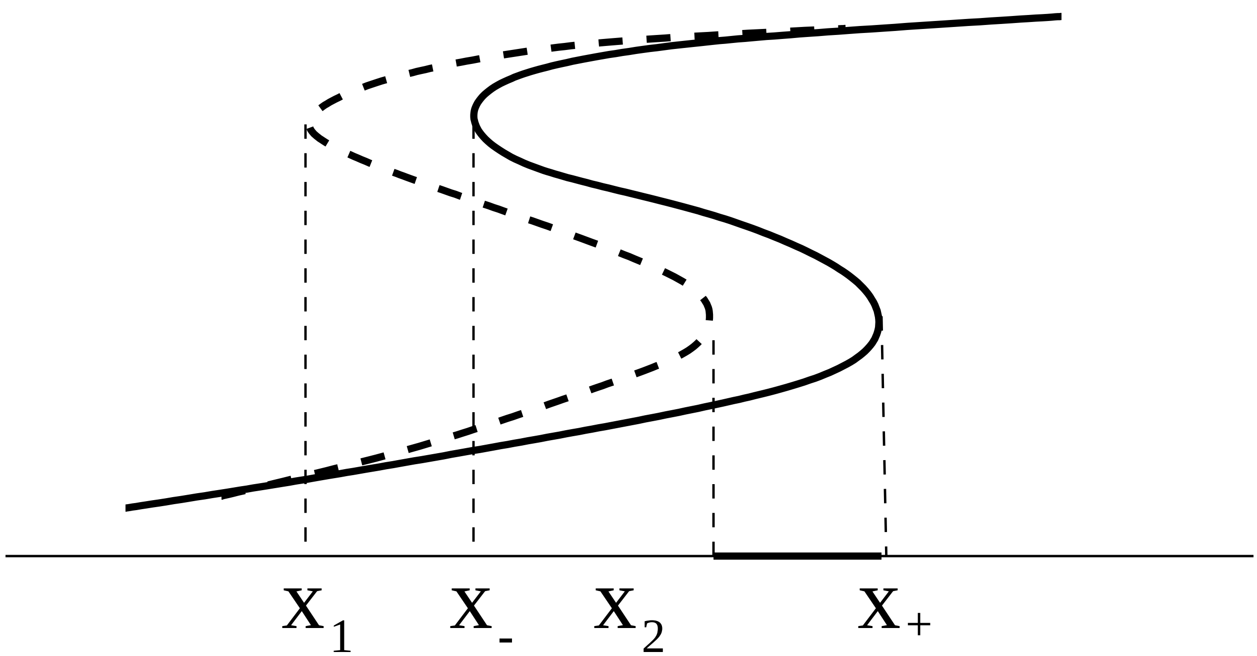

Fig. 1

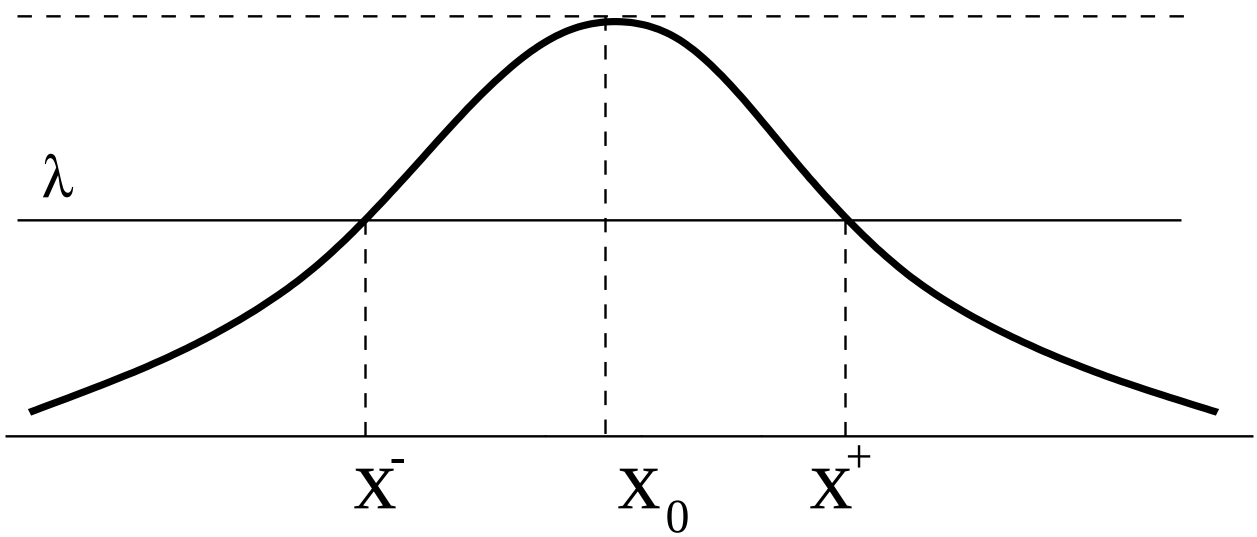

Fig. 2a

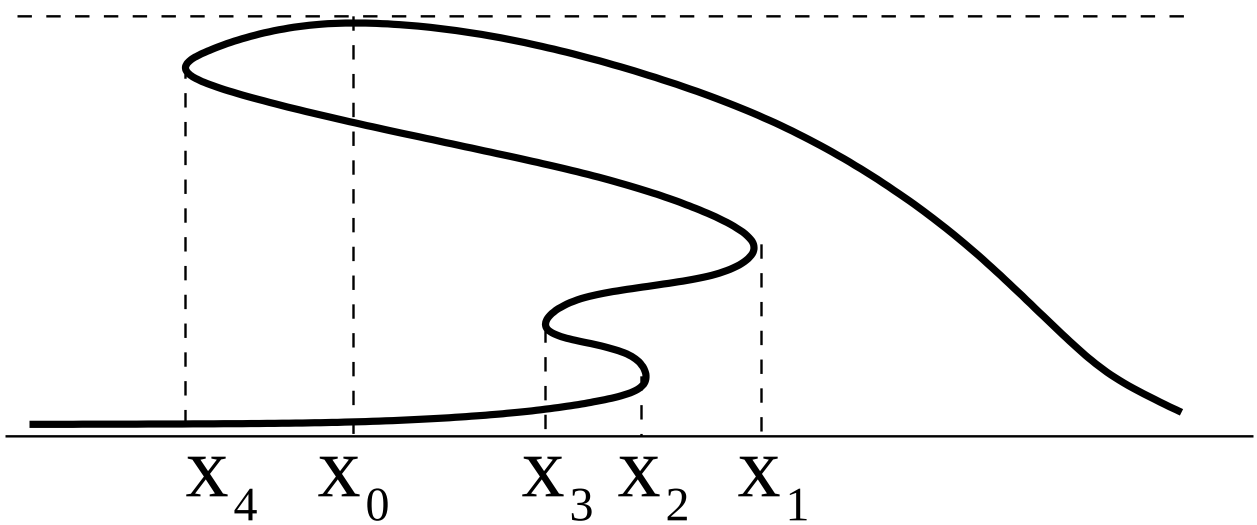

Fig. 2b

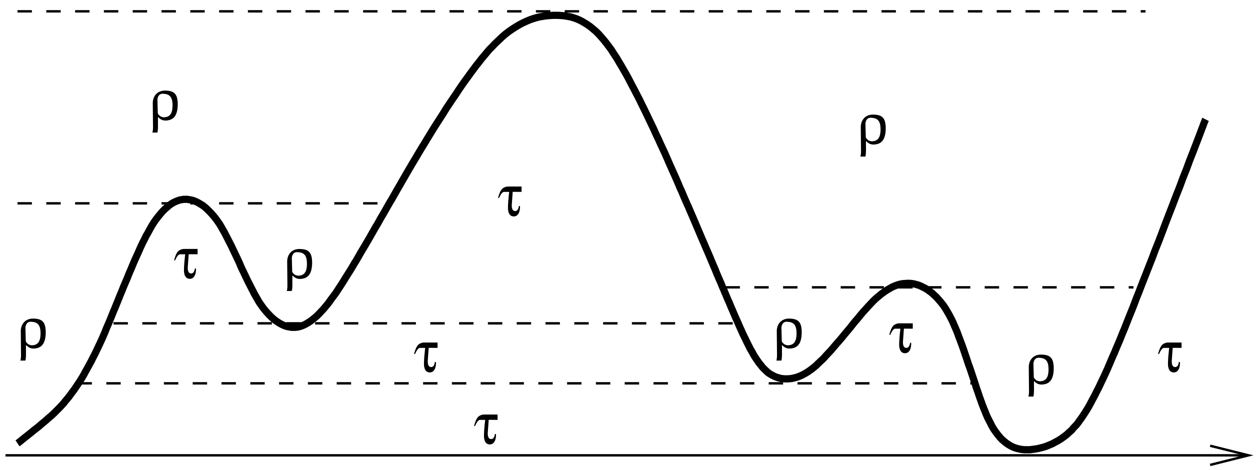

Fig. 3a

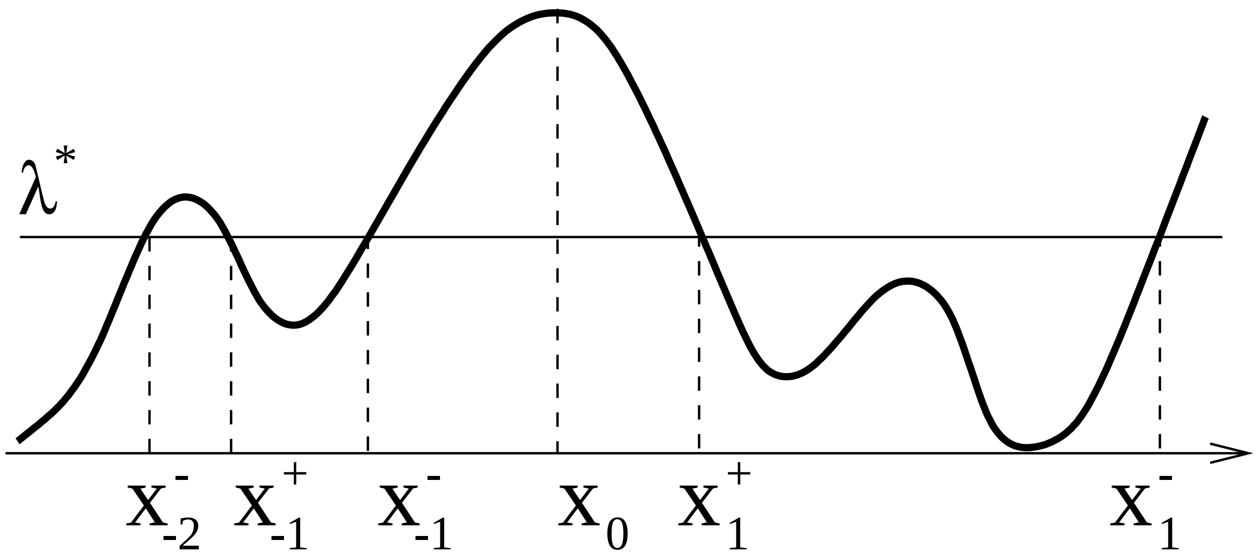

Fig. 3b